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# $J$-unitary equivalence of positive subspaces of a Krein space

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A $J$-unitary operator on a Krein space preserves the indefinite metric and hence maps positive subspaces to positive subspaces. The purpose of this note is to answer the following question: if $L, M$ are maximal positive subspaces of a Krein space, under what conditions does there exist a $J$-unitary operator $U$ such that $UL=M$? The answer turns out to be pleasantly concise. Note an obvious necessary condition: $L$ and $M$ must be isometrically isomorphic with respect to the semi-norms induced on them by the indefinite metric, and the same goes for their orthogonal companions. For most subspaces these conditions are also sufficient for the existence of the desired $U$, but there is an exceptional class for which an extra condition, related to the Fredholm index, is needed.

It is well known [3, 4] that questions on the action of $J$-unitaries on maximal positive subspaces can be re-formulated in terms of symplectic transformations of contraction operators, and in fact the theorems below are little more than a re-interpretation in the context of Krein spaces of recent results of the author's on orbits of the symplectic group [5].

Let $H$ be a Krein space with fundamental symmetry $J$. That is, $H$ is a Hilbert space with inner product $(\cdot, \cdot)$, having a distinguished orthogonal decomposition $H=H_+ \oplus H_-$, and $J=P_+ - P_-$ where $P_+, P_-$ are the orthogonal projections on $H_+, H_-$ respectively. We introduce an indefinite inner product $[\cdot, \cdot]$ on $H$ by $[x, y]= =(Jx, y)$, and we call an operator $U$ on $H$ *$J$-unitary* if $U$ is surjective and $[Ux, Uy]= =[x, y]$ for all $x, y \in H$, or equivalently, if

$$U^* J U = J = U J U^*. \tag{1}$$

A subspace $L$ of $H$ is said to be *positive* if $[x, x] \geqq 0$ for all $x \in L$, and to be *maximal positive* if it is positive and is not a proper subset of any positive subspace. Likewise one defines *negative* and *maximal negative* subspaces. If $L$ is any subspace of $H$ then

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the *orthogonal companion* $L^J$ of $L$ is the subspace

$$L^J = \{x \in H\colon [x, y] = 0 \quad \text{for all} \quad y \in L\}.$$

The orthogonal companion of a maximal positive subspace is a maximal negative subspace. If $L$ is a positive or negative subspace then the indefinite metric of $H$ induces a semi-norm $p$ on $L$ by $p(x) = |[x, x]|^{1/2}$; this will be called the *intrinsic semi-norm* of $L$.

If $U$ is $J$-unitary and $UL = M$ for some positive subspace $L$, then clearly $M$ is positive and $L, M$ are isometrically isomorphic with respect to their intrinsic semi-norms. Moreover, $U(L^J) = M^J$ and $L^J, M^J$ are also isometrically isomorphic. As was indicated above, this pair of isometric isomorphisms does not in general suffice for the existence of a $J$-unitary $U$ such that $UL = M$, and we are obliged to introduce another condition.

We shall say that a maximal positive subspace $L$ of $H$ is *Fredholm* if $P_-L$ is closed in $H_-$ and both $L \cap H_+$ and $L^J \cap H_-$ are finite-dimensional. We define the *signature* of a Fredholm positive subspace $L$ to be $\dim(L \cap H_+) - \dim(L^J \cap H_-)$.

Note that if $H$ is finite-dimensional then every positive subspace of $H$ is Fredholm and has signature $\dim H_+ - \dim H_-$, which is the signature of the Hermitian form $[x, x]$ in the classical sense of Sylvester.

We shall say that a topological vector space $E$ is *permanently incomplete* if it contains no complete infinite-dimensional subspace modulo the closure of $\{0\}$: that is, if $E/E_0$ with the quotient topology contains no complete infinite-dimensional subspace, where $E_0$ is the closure of $\{0\}$ in $E$.

Theorem. *Let $H$ be a separable Krein space with fundamental symmetry $J$ and let $L, M$ be maximal positive subspaces of $H$. There exists a $J$-unitary operator $U$ on $H$ such that $UL = M$ if and only if the following two conditions hold:*

(i) *$L, L^J$ are isometrically isomorphic to $M, M^J$ respectively in their intrinsic semi-norms, and*

(ii) *if $L$ and $L^J$ are permanently incomplete with respect to their intrinsic semi-norms then $L$ and $M$ have the same signature.*

It will transpire during the proof that if $L$ and $L^J$ are permanently incomplete then $L$ is Fredholm, so that condition (ii) makes sense.

We prove the theorem using the correspondence between positive subspaces and contractions. Every maximal positive subspace $L$ of $H$ is the graph of a contraction from $H_+$ to $H_-$; that is,

$$L = \{\langle x_+, Kx_+\rangle\colon x_+ \in H_+\} \tag{2}$$

where $K\colon H_+ \to H_-$ is a linear operator and $\|K\| \leq 1$. $K$ is uniquely determined by $L$ and is called the *angle operator* of $L$. It is clear that, for any contraction $K$ from $H_+$ to $H_-$, the space $L$ defined by (2) is a maximal positive subspace of $H$, so there is a one-one correspondence between maximal positive subspaces of $H$ and contractions from $H_+ \to H_-$. See [1, 3, or 4] for fuller details. If $L$ has angle operator $K$ then the maximal negative subspace $L^J$ is the graph of the contraction $K^*: H_- \to H_+$. Furthermore we have

$$P_- L = \operatorname{Range} K; \ \operatorname{Ker} K = L \cap H_+; \ \operatorname{Ker} K^* = L^J \cap H_-$$

and so $L$ is a Fredholm subspace in the sense defined above if and only if its angle operator $K$ is a Fredholm operator (see, for example [2]), and the signature of a Fredholm $L$ is the Fredholm index of its angle operator.

Applying a $J$-unitary transformation $U$ to a maximal positive subspace corresponds to taking a symplectic transformation of its angle operator. With respect to the orthogonal decomposition $H = H_+ \oplus H_-$ of $H$, $U$ can be written as a $2\times 2$ matrix of operators

$$U = \begin{bmatrix} D & C \\ B & A \end{bmatrix} \tag{3}$$

where $D: H_+ \to H_+$, $C: H_- \to H_+$ etc. are linear operators. It is easily calculated [4] that if $L$ is maximal positive with angle operator $K$ then $UL$ has angle operator

$$\Phi(K) = (AK+B)(CK+D)^{-1}. \tag{4}$$

A transformation of the form (4), where $U$ given by (3) is $J$-unitary on $H$, is called a *symplectic transformation* on the space $L(H_+, H_-)$ of all bounded linear operators from $H_+$ to $H_-$. The condition (1) (that $U$ be $J$-unitary) does ensure that $\Phi(K)$ be well defined — that is, $CK+D$ is invertible on $H_+$ for every contraction $K \in L(H_+, H_-)$ (see [3]).

In view of these correspondences we perceive that the spaces of the form $UL$, where $U$ is $J$-unitary, are those maximal positive subspaces of $H$ whose angle operators lie in the same orbit of the closed unit ball of $L(H_+, H_-)$ under the symplectic group as the angle operator $K$ of $L$. We can therefore make use of Theorem 1 of [5] which states the following, for separable Hilbert spaces $H_+, H_-$.

*Let $X, Y \in L(H_+, H_-)$ be contractions. There exists a symplectic transformation $\Phi$ on $L(H_+, H_-)$ such that $\Phi(X) = Y$ if and only if*

(S1) *$I - Y^*Y$, $I - YY^*$ are congruent to $I - X^*X$, $I - XX^*$ respectively, and*

(S2) *if $X$ is essentially unitary then* $\operatorname{ind} X = \operatorname{ind} Y$.

Here two Hermitian operators $M, N$ on a Hilbert space $G$ are said to be congruent if there exists an invertible operator $T$ on $G$, with bounded inverse, such that $M = T^*NT$. And an operator $X$ is said to be *essentially unitary* if $I - X^*X$ and $I - XX^*$ are compact operators.

Suppose, then, that $L, M$ are maximal positive subspaces of $H$ with angle operators $X, Y$ respectively. Our discussion shows that there exists a $J$-unitary operator

$U$ on $H$ such that $UL=M$ if and only if $X$ and $Y$ satisfy the conditions (S1) and (S2) above. It remains to translate these into statements about the geometry of $L$ and $M$.

Lemma 1. *$L$ and $M$ are isometrically isomorphic in their intrinsic semi-norms if and only if $I-Y^*Y$ is congruent to $I-X^*X$.*

Proof. ($\Rightarrow$) Let $R\colon L\to M$ be an isometric isomorphism. $R$ maps a typical element $\langle x_+, Xx_+\rangle$ of $L$ to an element $\langle y_+, Yy_+\rangle$ of $M$: write $y_+=Tx_+$. $T$ is clearly a bijective bounded linear operator on $H_+$, and so it is invertible, by the closed graph theorem. The intrinsic semi-norm $p$ on $L$ is given by

$$p(\langle x_+, Xx_+\rangle)^2 = (x_+, x_+)-(Xx_+, Xx_+) = ((I-X^*X)x_+, x_+),$$

while for the intrinsic semi-norm $q$ on $M$ we have

$$q(R\langle x_+, Xx_+\rangle)^2 = q(\langle Tx_+, YTx_+\rangle)^2 = ((I-Y^*Y)Tx_+, Tx_+).$$

Hence the supposition that $R$ be an isometry entails $I-X^*X=T^*(I-Y^*Y)T$. Conversely, if this congruence relation holds for some invertible $T$ then the formula $R\langle x_+, Xx_+\rangle=\langle Tx_+, YTx_+\rangle$ defines an isometric isomorphism $R\colon L\to M$.

Lemma 2. *Let $E$ be a Hilbert space and let $T$ be a bounded linear operator on $E$. Let $p_T(x)=\|Tx\|$ for $x\in E$. Then $E$ is permanently incomplete with respect to the semi-norm $p_T$ if and only if $T$ is compact.*

Proof. Let $F=E/\mathrm{Ker}\,T$ and let $K\colon E\to F$ be the quotient mapping. $p_T$ induces a norm on $F$, which we again denote by $p_T$. $T$ induces an operator $T_1\colon F\to E$ such that the diagram

$$\begin{tikzcd} E \arrow[rr, "K"] \arrow[dr, "T"'] & & (F, p_T) \arrow[dl, "T_1"] \\ & E & \end{tikzcd}$$

commutes. $T_1$ is an isometry of $F$ with Range $T$. It is easy to see (using the polar decomposition and the spectral theorem) that $T$ is compact if and only if Range $T$ contains no closed infinite-dimensional subspace, hence if and only if $(F, p_T)$ contains no complete infinite-dimensional subspace. This is the definition of the permanent incompleteness of $(E, p_T)$.

Corollary. *$X$ is essentially unitary if and only if both $L$ and $L^J$ are permanently incomplete in their intrinsic semi-norms.*

Proof. The proof of Lemma 1 shows that the intrinsic semi-norm $p$ of $L$ is given by

$$p(\langle x_+, Xx_+\rangle) = \|(I-X^*X)^{1/2}x_+\|.$$

Lemma 2 now shows that $(L, p)$ is permanently incomplete if and only if $(I-X^*X)^{1/2}$ is compact, and this is so if and only if $I-X^*X$ is compact. Likewise $L^J$ is permanently incomplete if and only if $I-XX^*$ is compact.

We can now conclude the proof of the Theorem. Lemma 1 shows that the angle operators $X, Y$ of $L, M$ satisfy condition (S1) if and only if $L, L^J$ are isometrically isomorphic to $M, M^J$ in their intrinsic semi-norms, which is condition (i) of the theorem. For condition (ii), the Corollary shows that $L$ and $L^J$ are permanently incomplete if and only if $X$ is essentially unitary, and in this case the requirement $\operatorname{ind} X = \operatorname{ind} Y$ of (S2) is manifestly the same as the equality of the signatures of $L$ and $M$.

It might be asked whether condition (ii) of the Theorem is really needed: it is conceivable that the isometric isomorphism condition (i) might imply the equality of the signatures of $L$ and $M$ in the permanently incomplete case. In fact it does not, as an example in [5, Section 5] shows. There exist contractions $X, Y$ on a separable Hilbert space $G$ satisfying the congruence conditions (S1) above, with $X, Y$ being compact perturbations of the identity and a unilateral shift respectively. They are thus essentially unitary but of different index. If we define a Krein space $H$ with $H_+ = G = H_-$ and take $L, M$ to be the maximal positive subspaces with angle operators $X, Y$ respectively then $L, M$ satisfy condition (i) but not condition (ii) of the Theorem.

We note that Theorem 2 of [5] gives a recipe for constructing all symplectic transformations $\Phi$ such that $\Phi(X) = Y$ in terms of $X, Y$ and the operators implementing the congruences of condition (i) of the Theorem.

## References

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